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Towards Finding Hay in a Haystack: Explicit Tensors of Border Rank Greater Than $2.02m$ in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$

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Abstract. We write down an explicit sequence of tensors in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$, for all m sufficiently large, having border rank at least $2.02m$, overcoming a longstanding barrier. We obtain our lower bounds via the border substitution method.

1 Introduction

A frequently occurring theme in algebraic complexity theory is the *hay in a haystack* problem (phrase due to Howard Karloff): find an explicit object that behaves generically. For example Valiant's **VP** vs. **VNP** problem is to find an explicit polynomial sequence that is difficult to compute. We address this problem for three-way tensors. Here the state of the art is embarrassing.

Let A, B, C be complex vector spaces of respective dimensions $\mathbf{a}, \mathbf{b}, \mathbf{c}$. A tensor $T \in A \otimes B \otimes C$ has *rank one* if $T = a \otimes b \otimes c$ for some $a \in A$, $b \in B$, $c \in C$. The *rank* of T, denoted $\mathbf{R}(T)$, is the

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smallest r such that T is a sum of r rank one tensors. The *border rank* of T, denoted $\underline{\mathbf{R}}(T)$, is the smallest r such that T is the limit of a sequence of rank r tensors.

If a tensor $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ is chosen randomly with $m > 3$, then with probability one, it will have rank and border rank $\lceil \frac{m^3}{3m-2} \rceil \sim \frac{m^2}{3}$ [10] [1]. More precisely, the set of tensors with border rank less than $\lceil \frac{m^3}{3m-2} \rceil$ is a proper subvariety and the set of tensors with rank not equal to $\lceil \frac{m^3}{3m-2} \rceil$ is contained in a proper subvariety. Previous to this paper, there was no explicit sequence of tensors of border rank at least $2m$ known (and no explicit sequence of rank at least $3m$ known), although several known sequences come close to these. In [1] they exhibit an explicit sequence of tensors satisfying $\mathbf{R}(T_m) \geq 3m - O(\log_2(m))$ and in [5] a sequence of tensors of border rank at least $2m-2$ is presented, although these tensors are only "mathematician explicit" and not "computer-scientist explicit" as some entries of the tensor are of the form 2^{2^m}. We explain the notion of explicitness in Section 3. The best border rank lower bound for a computer-scientist-explicit tensor to our knowledge is ironically the tensor $M_{\langle \mathbf{n} \rangle}$ corresponding to matrix multiplication of $\mathbf{n} \times \mathbf{n}$ matrices where, setting $m = \mathbf{n}^2$, $\underline{\mathbf{R}}(M_{\langle \mathbf{n} \rangle}) \geq 2m - \lceil \frac{1}{2} \log_2(m) \rceil - 1$ [8]. One of the central problems is that the typical way to prove that a tensor T has high border rank is to determine a polynomial P that vanishes on each tensor of border rank r and prove that $P(T) \neq 0$. This shows that the border rank of T is strictly greater than r. Still, in spite of many results (see, e. g., [6] and references therein), there are no known polynomials on the space $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ that vanish on tensors of border rank $2m$. For this reason the bound $2m$ remained unbroken until now. Here, instead of finding new polynomials vanishing on tensors of high rank, we combine several methods: border substitution, Koszul flattenings (which are polynomials), combinatorics of normal forms, degenerations as well as explicit computational techniques. In this paper we deal exclusively with border rank.

Definition 1.1. A sequence of objects (e. g., numbers, graphs, tensors, typically each represented as a finite sequence of integers), indexed by integers, is *explicit* if the n-th object may be computed in time polynomial in n.

We remark that there are more restrictive definitions of what it means for a sequence to be explicit but the experts we asked viewed the above one as acceptable.

Thus, a sequence $T_m \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ of tensors is explicit if the entries of T_m may be computed in time polynomial in m. This is equivalent to saying that all of T_m may be computed in polynomial time in m.

We show:

Theorem 1.2. *For each k, define the $\lceil \frac{3(2k+1)^2}{4} \rceil$-dimensional family of tensors $T_k = T_k(p_{ij}) \in \mathbb{C}^{2k+1} \otimes$*

[1]The conclusion holds with almost any notion of randomness, e. g., view the ambient $\mathbb{C}^{m^3}$ as a vector space with any measure that is absolutely continuous with respect to Lebesgue measure. Then the set of tensors for which the equality fails has measure zero. If instead one takes the entries of T from a finite subset of the integers with a uniform distribution, then the probability of the equality failing goes to zero as the subset one draws from increases towards all of $\mathbb{Z}$.

$\mathbb{C}^{2k+1} \otimes \mathbb{C}^{2k+1} =: A \otimes B \otimes C$ as follows:

$$T_k = \sum_{i=-k}^{k} \sum_{j=\max(0,-i)}^{\min(2k-i,2k)} p_{ij} a_i \otimes b_j \otimes c_{i+j} , \tag{1.1}$$

where $A \simeq B \simeq C \simeq \mathbb{C}^{2k+1}$ and $(a_{-k}, \ldots, a_k), (b_0, \ldots, b_{2k}), (c_0, \ldots, c_{2k})$ are bases of the vector spaces A, B, C, respectively.

Then for all

$$0 < \epsilon < \frac{1}{42}, \text{ and } k \geq \frac{7(52274 - 151635\epsilon + 14616\epsilon^2 + 5292\epsilon^3)}{(1 - 42\epsilon)^3} ,$$

we describe an explicit assignment of the p_{ij} in the sense of Definition 1.1, so that

$$\underline{\mathbf{R}}(T_k) \geq (2 + \epsilon)(2k + 1) = (2 + \epsilon)m .$$

In particular, when $\epsilon = .001$ we need $k \geq 414797$, and when $\epsilon = .02$ we need $k \geq 84162677$.

Viewing T_k as a matrix of linear forms, e. g., $T(A^*)$, the matrix is zero in the upper right and lower left corners, e. g., when $k = 2$,

$$T_2(A^*) = \begin{pmatrix} * & * & * & 0 & 0 \\ * & * & * & * & 0 \\ * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix} .$$

The explicit assignment is as follows: for $|i| > \frac{3}{7}k$, list the p_{ij} in some order and assign them distinct prime numbers as in Lemma 3.1. For $|i| \leq \frac{3}{7}k$, assign them periodically as in Lemma 3.2. More precisely, set $K = \lceil \frac{78}{1-42\epsilon} \rceil$. Then the value of p_{ij} depends only on i mod $4K$ and j mod $4K$. For $0 \leq i, j < 4K$ the values of p_{ij} are determined in such a way that distinct values are never coordinates of a solution to any of the explicit nonzero polynomials that we derive in Lemma 4.3.

If k is smaller, we can still get border rank bounds of twice the dimension or greater. For example:

Theorem 1.3. *Let T_6 be assigned the values $p_{ij} = 2^i + 3^j - 7ij$. Then $\underline{\mathbf{R}}(T_6) \geq 26 = 2m$.*

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2 Background

Definition 2.1. Let $T \in A \otimes B \otimes C$ be a tensor and let $A^*_f \subset A^*$ be a linear subspace. The *restriction* of T to A^*_f, denoted $T|_{A^*_f \otimes B^* \otimes C^*} =: T|_{A^*_f}$ is obtained by identifying T with a linear map $A^* \to B \otimes C$ and restricting the domain to the subspace A^*_f.

A tensor $T \in A \otimes B \otimes C$ is *A-concise* if the induced map $A^* \to B \otimes C$ is injective. Intuitively, T cannot be realized as a tensor in some $\mathbb{C}^{\mathbf{a}-1} \otimes B \otimes C$.

The *border substitution method* of [7] begins with the observation that, for each fixed $f < \mathbf{a}$ and A-concise tensor T,

$$\underline{\mathbf{R}}(T) \geq \min_{A^*_f \subset A^*} \left(\mathbf{a} - f + \underline{\mathbf{R}}(T|_{A^*_f \otimes B^* \otimes C^*}) \right) \tag{2.1}$$

where the min is over all subspaces A^*_f of A^* of dimension f. As stated, this is nearly useless as it is not practical to test all f-dimensional subspaces of A^*. However, when a tensor has symmetry, the normal form lemma [7, Lem. 3.1] allows one to restrict the search to a smaller subset. In particular, when it is invariant under a torus, i. e., a continuous abelian diagonalizable subgroup, it is sufficient to test the A^*_f that are invariant under the action of the torus. As observed by Strassen (and is well known among algebraic geometers), torus invariant objects can be studied via combinatorics. In our situation, the relevant property is as follows:

For a tensor $T = \sum_{ijk} t^{ijk} a_i \otimes b_j \otimes c_k$ given in bases, the *support* of T is $\operatorname{supp}(T) := \{(ijk) \mid t^{ijk} \neq 0\}$.

Definition 2.2 ([13]). A tensor $T \in A \otimes B \otimes C$ is *tight* if there exist bases of A, B, C and injective functions $\tau_A : [\mathbf{a}] \to \mathbb{Z}$, $\tau_B : [\mathbf{b}] \to \mathbb{Z}$ and $\tau_C : [\mathbf{c}] \to \mathbb{Z}$ such that for all $(i, j, k) \in \operatorname{supp}(T)$, $\tau_A(i) + \tau_B(j) + \tau_C(k) = 0$. (See [2] for details.) Call such a basis a *tight basis*.

Given a tight basis, the tensor T is invariant with respect to the $\mathbb{C}\backslash\{0\}$-action where $t \in \mathbb{C}\backslash\{0\}$ acts on the i-th, j-th, and k-th basis vector of A, B, and C by multiplication by $t^{\tau_A(i)}$ $t^{\tau_B(j)}$, and $t^{\tau_C(k)}$, respectively.

The tensor (1.1) is A-concise (for the given assignment of the p_{ij} and tight with $\tau_A(i) = i$, $\tau_B(j) = j$, $\tau_C(k) = k$.

The border substitution method and normal form lemma of [7] imply:

Proposition 2.3. *Let $T \in A \otimes B \otimes C$ be tight, A-concise, and expressed in a tight basis. Then among the filtrations $A^*_1 \subset \cdots \subset A^*_{\mathbf{a}-1} \subset A^*$ of A^*, i. e., among nested sequences of subspaces where $\dim(A^*_f) = f$, where each A^*_f is spanned by tight basis elements, there is at least one such that*

$$\underline{\mathbf{R}}(T) \geq \max_q (q + \underline{\mathbf{R}}(T(q))) ,$$

where $T(q) := T|_{A^*_{\mathbf{a}-q} \otimes B^* \otimes C^*}$.

For convenience of the reader, we give a self-contained proof:

Proof. By induction on $\dim A^*$ it is enough to prove that we may choose $A^*_{\mathbf{a}-1}$ as in the proposition. Fix $f := \mathbf{a}-1$. For any $g \in GL(A)$, if we replace T by gT, and A^*_f by gA^*_f in Eq. (2.1) the inequality still holds. The Borel fixed point theorem (see, e. g., [4, §21.2]) says that a connected solvable group acting on a complete variety X has a fixed point in X. In our case the connected solvable group is the torus $\mathbb{T} = \mathbb{C}\backslash\{0\}$ and X is the $\mathbb{T}$-orbit closure of some $A^*_{f,0} \in G(f, A^*) = \mathbb{P}(A)$ achieving the minimum in Eq. (2.1). Since $gT = T$ for all $g \in \mathbb{T}$, the right hand side of Eq. (2.1) is unchanged. Finally the $\mathbb{T}$-fixed points are exactly the f-planes spanned by basis vectors. □

Thus tight tensors are a natural class of tensors where the border substitution method may be effectively implemented because there are only a finite number of filtrations to check. The family T_k of Theorem 1.2 is tight.

If T is tight, then any tensor with the same support as T in a tight basis is also tight. In particular, it is natural to study tight tensors in families indexed by the support. In this context, it was shown in [2] that the largest possible support for a tight tensor is given by the family (1.1), which is a $\lceil \frac{3}{4}m^2 \rceil$-dimensional family, where $m = 2k+1$. That is, the family (1.1) is the largest such that Proposition 2.3 may be applied to.

The idea of the proof of Theorem 1.2 is as follows: think of a game where we are trying to prove a lower bound and an adversary is trying to prevent it by zeroing out basis vectors of T_k in a way that lowers the border rank of the degenerated tensor as much as possible. We show that, no matter what strategy the adversary chooses, there is a q where we obtain the desired lower bound using the Koszul flattenings described in Section 4 by further degenerating the already degenerated tensor to make the lower bound estimate from the Koszul flattenings transparent. After describing Koszul flattenings, in Section 5 we group the adversary's possible strategies into three types, and show that for each strategy we can choose a q and a further degeneration to obtain our bound.

3 Explicit tensors

The following two types of sequences of tensors are explicit in the sense of Definition 1.1.

Lemma 3.1. *Let $T_m \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ be a sequence of tensors whose only nonzero entries are as in tensor (1.1). If the entries of T_m are the first distinct m^2 prime numbers, then T_m is explicit.*

Proof. Let $\pi(x)$ be the number of prime numbers smaller or equal than x. By [11] we know that $\frac{x}{\log x} < \pi(x)$ for $x \geq 17$. Hence, running the sieve method on first m^3 numbers, we may find the first m^2 prime numbers in polynomial time. □

The following is obvious but we record it for future use:

Lemma 3.2. *Let $T_m \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ be a sequence of tensors whose only nonzero entries are as in tensor (1.1). If the entries of T_m are taken from a fixed library $\{c_{a,b}\}_{0\leq a<\mathcal{A}, 0\leq b<\mathcal{B}}$ of integers periodically, so $p_{ij} = c_{(i \bmod \mathcal{A}),(j \bmod \mathcal{B})}$), then the sequence T_m is explicit.*

The tensors in our sequence will be a sum of tensors of the types from Lemmas 3.1 and 3.2, and thus also explicit.

4 Koszul flattenings and border rank lower bounds

We first review Koszul flattenings. We then apply Koszul flattenings to a projection of the tensor T_6 and two general types of tensors, each of which arises as a degeneration of $T_{k,q} := T_k|_{A^*_{2k+1-q}\otimes B^*\otimes C^*}$.

4.1 Koszul flattenings

Fix an integer $p < \frac{1}{2}\dim A$. Fix bases $\{a_i\}$, $\{b_j\}$, $\{c_k\}$ of A, B, C. Let $T = \sum_{ijk} t^{ijk} a_i \otimes b_j \otimes c_k \in A \otimes B \otimes C$. Consider the linear map

$$\begin{aligned} T_A^{\wedge p} : \Lambda^p A \otimes B^* &\to \Lambda^{p+1} A \otimes C \\ X \otimes \beta &\mapsto \sum_{ijk} t^{ijk} \beta(b_j)(a_i \wedge X) \otimes c_k. \end{aligned}$$

Then [9, Proposition 4.1.1] states

$$\underline{\mathbf{R}}(T) \geq \frac{\operatorname{rank}(T_A^{\wedge p})}{\binom{\dim(A)-1}{p}}. \tag{4.1}$$

Let $A'^* \subset A^*$ be a subspace. Write $\phi : A \to A/(A'^*)^{\perp} =: A'$ for the projection onto the quotient. Here $(A'^*)^{\perp} \subset A$ is the annihilator of $A'^* \subset A^*$. The corresponding Koszul flattening map gives a lower bound for $\underline{\mathbf{R}}(\phi(T))$, which, by linearity, is a lower bound for $\underline{\mathbf{R}}(T)$.

One often takes a subspace $A'^* \subseteq A^*$ of dimension $2p + 1$ in the restriction of T, so the denominator $\binom{\dim(A)-1}{p}$ is replaced by $\binom{2p}{p}$ in Eq. (4.1).

The $p = 1$ case is a generalization of Strassen's equations [12], and for tensors in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ it can at best prove border rank lower bounds of $\frac{3}{2}m$. The larger the p one takes, Koszul flattenings become closer to potentially being able to prove a border rank lower bound of $2m$. However, the equations become more difficult to implement. We utilize the $p = 2$ case, which can at best prove border rank lower bounds of $\frac{5}{3}m$. This case can be reduced to lower bounding a matrix whose entries are at worst cubic in the p_{ij} of tensor (1.1). Thus, before using Koszul flattenings, we will need to use the border substitution method to kill off at least a $(\frac{1}{3} + \epsilon)m$-dimensional space. In our case we will kill off a space of dimension $\sim \frac{1}{2}m$ because our Koszul flattening matrices will not be of maximal rank.

Let $T \in \mathbb{C}^5 \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ and write $T = a_1 \otimes X_1 + \cdots + a_5 \otimes X_5$. Order the bases of $\bigwedge^2 \mathbb{C}^5$, $a_i \wedge a_{i'}$, with $ii' = 12, 13, 14, 15, 23, 24, 25, 34, 35, 45$ and of $\bigwedge^3 \mathbb{C}^5$ as $234, 235, 245, 345, 123, 124, 125, 134, 135, 145$.

Then the $10n \times 10n$ Koszul flattening matrix takes the block form

$$T_A^{\wedge 2} = \begin{array}{c} \begin{matrix} 12 & 13 & 14 & 15 & 23 & 24 & 25 & 34 & 35 & 45 \end{matrix} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & X_4 & -X_3 & 0 & X_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & X_5 & 0 & -X_3 & 0 & X_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & X_5 & -X_4 & 0 & 0 & X_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & X_5 & -X_4 & X_3 \\ X_3 & -X_2 & 0 & 0 & X_1 & 0 & 0 & 0 & 0 & 0 \\ X_4 & 0 & -X_2 & 0 & 0 & X_1 & 0 & 0 & 0 & 0 \\ X_5 & 0 & 0 & -X_2 & 0 & 0 & X_1 & 0 & 0 & 0 \\ 0 & X_4 & -X_3 & 0 & 0 & 0 & 0 & X_1 & 0 & 0 \\ 0 & X_5 & 0 & -X_3 & 0 & 0 & 0 & 0 & X_1 & 0 \\ 0 & 0 & X_5 & -X_4 & 0 & 0 & 0 & 0 & 0 & X_1 \end{pmatrix} \end{array} \begin{matrix} \\ 234 \\ 235 \\ 245 \\ 345 \\ 123 \\ 124 \\ 125 \\ 134 \\ 135 \\ 145 \end{matrix} . \tag{4.2}$$

If X_1 is invertible then the rank of the matrix (4.2) equals $6n$ plus the rank of the block $4n \times 4n$ matrix with block entries

$$\begin{pmatrix} X_3X_1^{-1}X_4 - X_4X_1^{-1}X_3 & X_4X_1^{-1}X_2 - X_2X_1^{-1}X_4 & X_2X_1^{-1}X_3 - X_3X_1^{-1}X_2 & 0 \\ X_3X_1^{-1}X_5 - X_5X_1^{-1}X_3 & X_5X_1^{-1}X_2 - X_2X_1^{-1}X_5 & 0 & X_2X_1^{-1}X_3 - X_3X_1^{-1}X_2 \\ X_4X_1^{-1}X_5 - X_5X_1^{-1}X_4 & 0 & X_5X_1^{-1}X_2 - X_2X_1^{-1}X_5 & X_2X_1^{-1}X_4 - X_4X_1^{-1}X_2 \\ 0 & X_4X_1^{-1}X_5 - X_5X_1^{-1}X_4 & X_5X_1^{-1}X_3 - X_3X_1^{-1}X_5 & X_3X_1^{-1}X_4 - X_4X_1^{-1}X_3 \end{pmatrix} . \tag{4.3}$$

To see this use the basic identity, for a block matrix with Z square and invertible:

$$\begin{pmatrix} 0 & X \\ Y & Z \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & XZ^{-1} \\ 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} -XZ^{-1}Y & 0 \\ 0 & Z \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ Z^{-1}Y & \mathrm{Id} \end{pmatrix} .$$

In what follows it will be more convenient to permute the blocks of the matrix (4.3) and multiply rows and columns by (-1), so that the matrix becomes block skew-symmetric:

$$\begin{pmatrix} 0 & -X_2X_1^{-1}X_3 + X_3X_1^{-1}X_2 & X_4X_1^{-1}X_2 - X_2X_1^{-1}X_4 & -X_3X_1^{-1}X_4 + X_4X_1^{-1}X_3 \\ X_2X_1^{-1}X_3 - X_3X_1^{-1}X_2 & 0 & -X_5X_1^{-1}X_2 + X_2X_1^{-1}X_5 & X_3X_1^{-1}X_5 - X_5X_1^{-1}X_3 \\ X_2X_1^{-1}X_4 - X_4X_1^{-1}X_2 & X_5X_1^{-1}X_2 - X_2X_1^{-1}X_5 & 0 & X_4X_1^{-1}X_5 - X_5X_1^{-1}X_4 \\ X_3X_1^{-1}X_4 - X_4X_1^{-1}X_3 & X_5X_1^{-1}X_3 - X_3X_1^{-1}X_5 & -X_4X_1^{-1}X_5 + X_5X_1^{-1}X_4 & 0 \end{pmatrix} . \tag{4.4}$$

We record the above observation.

Proposition 4.1. *Let $T = a_1 \otimes X_1 + \cdots + a_5 \otimes X_5 \in \mathbb{C}^5 \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ and let X_1 be of rank n. If the rank of the $4n \times 4n$ block matrix (4.3) is at least R, then $\mathrm{rank}(T_A^{\wedge 2}) \geq R + 6n$.*

4.2 Proof of Theorem 1.3

We first project T_6 to $\mathbb{C}^9 \otimes \mathbb{C}^{13} \otimes \mathbb{C}^{13}$ by considering all possible coordinate projections. By Proposition 2.3 for one of the projections the border rank drops at least by four. In the algorithm used in Section 6, all possible $\binom{13}{4}$ projections are considered.

Next, in each case we consider a general projection to $\mathbb{C}^5 \otimes \mathbb{C}^{13} \otimes \mathbb{C}^{13}$. We note that we do not have to prove that the chosen projection is general, as it is enough to find one projection. Finally we apply $p = 2$ Koszul flattenings described in Section 4.1. The algorithm, implemented in Macaulay 2 [3], checks that the rank of the matrix (4.2) is always at least 127. Thus:

$$\underline{\mathbf{R}}(T_6) \geq \lceil \frac{127}{6} \rceil + 4 = 22 + 4 = 26\,.$$

□

4.3 Koszul flattenings applied to a tensor with $T(A^*)$ supported on the diagonal and four shifted diagonals close to the diagonal

Recall that a Zariski open subset of a vector space is the complement of the zero set of a finite collection of polynomials on the space. In particular, given a fixed nonempty open subset, a random element of a vector space will lie in the subset with probability one.

Definition 4.2. A shifted diagonal of a matrix (m_{ij}) at distance y below (or above) the diagonal is the set of those entries where $i - j = y$ ($j - i = y$, resp.). A diagonal corresponds to the case $i = j$. We call a matrix a shifted-diagonal if it is supported on a shifted diagonal.

Lemma 4.3. *Let $T = a_1 \otimes X_1 + \cdots + a_5 \otimes X_5 \in \mathbb{C}^5 \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ be a tensor such that X_1 is diagonal with nonzero entries on the diagonal, X_2, X_3 are shifted-diagonal matrices supported on shifted diagonals at distances $x_2 < x_3$ above the main diagonal and X_4, X_5 are shifted-diagonal matrices supported on shifted diagonals $x_4 < x_5$ below the main diagonal, and assume $x_s < C$ for $s = 2, \ldots, 5$ for some constant C.*

Fixing one such T as above, the $p = 2$ Koszul flattening matrix drops rank by at most $16C$ on a Zariski open subset of tensors with the same shape as T. In particular, for most tensors T of that shape, $\underline{\mathbf{R}}(T) \geq \frac{5}{3}n - \frac{8}{3}C$.

Proof. As X_1 is invertible, the second Koszul flattening has a kernel of dimension equal to the dimension of the kernel of the matrix (4.3).

Write $M_{ts} := X_t X_1^{-1} X_s - X_s X_1^{-1} X_t$, $t < s$, for the six distinct size $n \times n$ block matrices appearing in the matrix (4.3) (here $M_{st} = -M_{ts}$). The M_{ts} are shifted-diagonal matrices. Explicitly, M_{23} is supported on shifted diagonal $x_2 + x_3$, M_{24} is supported on shifted diagonal $x_2 - x_4$, M_{25} is supported on shifted diagonal $x_2 - x_5$, M_{34} is supported on shifted diagonal $x_3 - x_4$, M_{35} is supported on shifted diagonal $x_3 - x_5$, M_{45} is supported on shifted diagonal $-(x_4 + x_5)$, where negative indices mean the shifted diagonal is below the main diagonal. We note that some of the entries on these shifted diagonals may be equal to zero.

Permuting the basis vectors, one obtains a block diagonal matrix, where the blocks are of size 4×4, namely, using cyclic notation, we group columns $x_5 + j$, $x_4 + j$, $-x_3 + j$, $-x_2 + j$ and rows $j - x_2 - x_3 + x_4$, $j - x_2 - x_3 + x_5$, $j - x_2 + x_4 + x_5$, $j - x_3 + x_4 + x_5$, and there are n such blocks, as in Figure 1. Call the resulting 4×4 blocks N^j, $1 \leq j \leq n$.

It remains to show that almost all matrices N^j have full rank. We note that each of the matrices M_{ts} has at most $4C$ entries on the distinguished shifted diagonal that are not a binomial in the entries of the X_i. These non-binomial entries appear only if one of the column or row indices of M_{ts} is either smaller than $2C$ or greater than $n-2C$. We focus on the N^j contained in the other rows and columns, i. e., $2C < j < n-2C$. A direct computation shows that the determinant of each such N^j is, after clearing denominators, an explicit nonzero polynomial P^j in the entries of the matrices X_i. We note that in the given range of indices the polynomials P^j are evaluations of one polynomial, but in different sets of variables.

Hence, whenever the evaluations of the $n-4C$ polynomials P^j are nonzero the bounds on ranks given in Lemma 4.3 hold. □

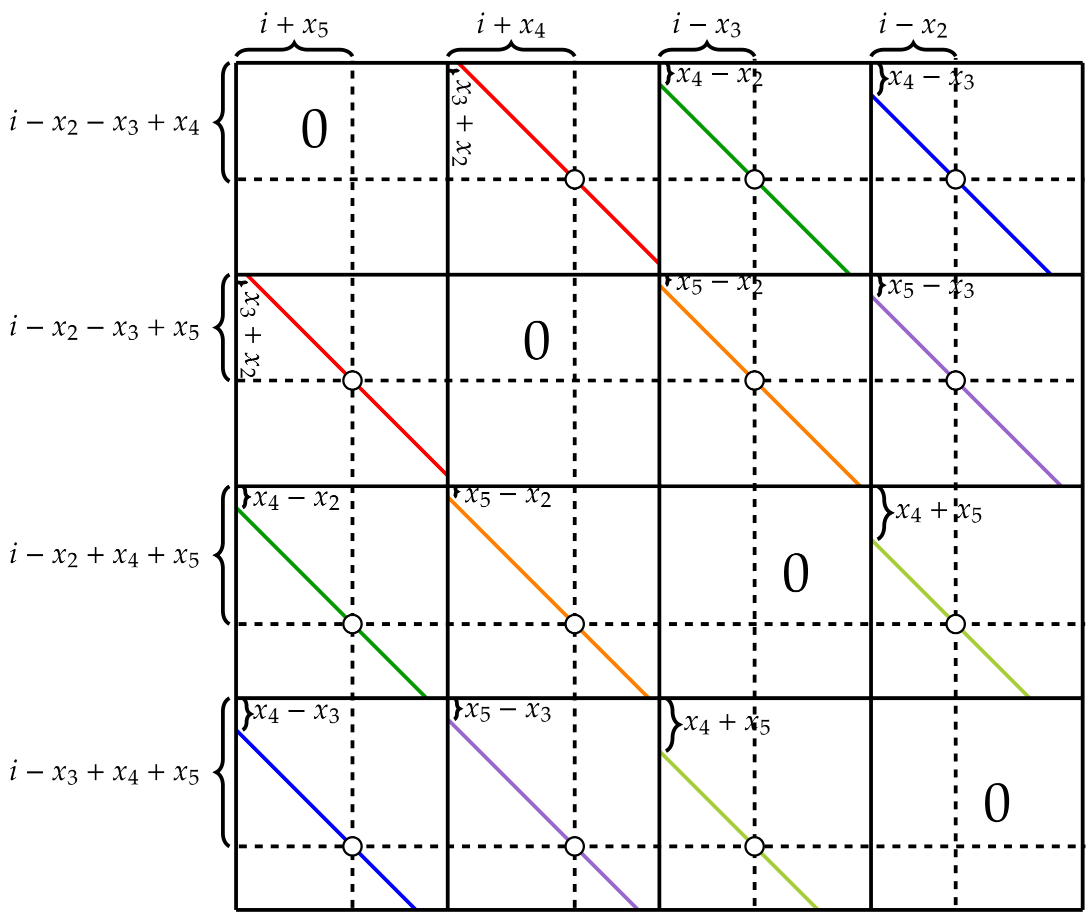

Figure 1: Matrix (4.4) for a tensor as in Lemma 4.3 after permuting basis vectors as in the proof. The i-th 4×4 block N^i has entries corresponding to the intersection of the dashed lines

Corollary 4.4. *Let T be a tensor as in Lemma 4.3 that is such a subtensor*[2] *of the tensor T_k that all X_i correspond to shifted diagonals of T of distance at most $\frac{3}{7}k$ to the main diagonal*[3]*. Then the assertion of*

[2]A *subtensor* is a tensor defined by the same formula as T_k, where we only sum over subsets of indices

[3]I. e., we only take a_i such that $|i| \le \frac{3}{7}k$ in Equation (1.1)

the Lemma 4.3 holds.

Proof. We keep the notation of Lemma 4.3 and its proof.

We note that each P^j, irrespective of the choice of T and j, depends only on variables corresponding to entries that are contained in a $4C \times 4C \times 4C$ subtensor of T_k made of consecutive indices, i. e., $(T_k)_{i,j,l}$, where $i \in \{N_1 + 1, N_1 + C\}$, $j \in \{N_2 + 1, N_2 + 4C\}$ and $l \in \{N_3 + 1, N_3 + C\}$ for some integers N_1, N_2, N_3. As C is a fixed constant and the P^j are explicit polynomials, the non-vanishing can be achieved by filling one $4C \times 4C \times 4C$ cube with numbers that do not form coordinates of any root of any of the P^j and then assigning the values of the tensor periodically as in Lemma 3.2. □

4.4 Koszul flattenings applied to a tensor supported on 5 shifted diagonals not equal to the diagonal

Lemma 4.5. *Let $T \in \mathbb{C}^5 \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ be a tensor supported on three shifted diagonals at distances $y_3 < y_4 < y_5$ below the diagonal and two shifted diagonals at distances $y_1 > y_2$ above the diagonal such that $y_1 - y_2 < y_3$ and $y_1 + y_5 < n$. Assume the entries of T are distinct primes. Then*

$$\underline{\mathbf{R}}(T) \geq n + y_1 - \frac{2}{3} y_2 \,.$$

Proof. Consider the second Koszul flattening matrix of T given in Eq. (4.2). The bound of $\underline{\mathbf{R}}(T)$ will follow by estimating the rank of this matrix.

Consider, in each of the last six blocks, the last $n - y_1$ columns. We obtain $6n - 6y_1$ independent columns due to the matrix X_1 in Eq. (4.2).

We now restrict our attention to columns that are still in last six blocks, but just consider the first y_1 columns in each block. We focus on the upper four horizontal blocks, obtaining a $4n \times 6y_1$ matrix, as in Figure 2. Note that in order for the lines in each block to have width y_1 we need $y_1 + y_5 < n$. We claim that the rank of this matrix is at least $4y_1 + 2(y_1 - y_2)$.

In this matrix the second, third and sixth vertical blocks give independent columns, due to the matrices that are y_3 below the diagonal (blue in Figure 2). We claim that the y_1 columns in the first vertical block are also independent of them. For contradiction suppose there is a linear relation involving i-th column in the first block. The $(i + y_4)$ entry of the first block of this column may only be canceled using the $(i + y_4 - y_3)$ column in the second block. Then the contribution in the third block may only be canceled by the $(i + y_5 - y_3)$ column in the third vertical block. Now the entries in the second horizontal block cancel if and only if a binomial cubic equation on the entries is satisfied. This is not possible, as the entries are distinct primes.

So far we obtain the lower bound on the rank of matrix in Figure 2 equal to $4y_1$. By adding the last $(y_1 - y_2)$ columns of fourth and fifth block, which are independent as $y_1 - y_2 < y_3$, we may find $2(y_1 - y_2)$ more independent columns.

We now focus on the first four vertical blocks in the matrix (4.2). Among those, the first four horizontal blocks are zero, thus we restrict to the last six. We further restrict to the rows that are zero in last six vertical blocks. We obtain a 6×4 block matrix, which is similar to reflection of the matrix in Figure 2. We lower bound the rank of this matrix in the same way as before.

In total we obtain that the rank of the matrix (4.2) is at least

$$6(n - y_1) + 2(4y_1 + 2(y_1 - y_2)) = 6n + 2(3y_1 - 2y_2).$$

This translates to the bound on the border rank as in the statement of the lemma. □

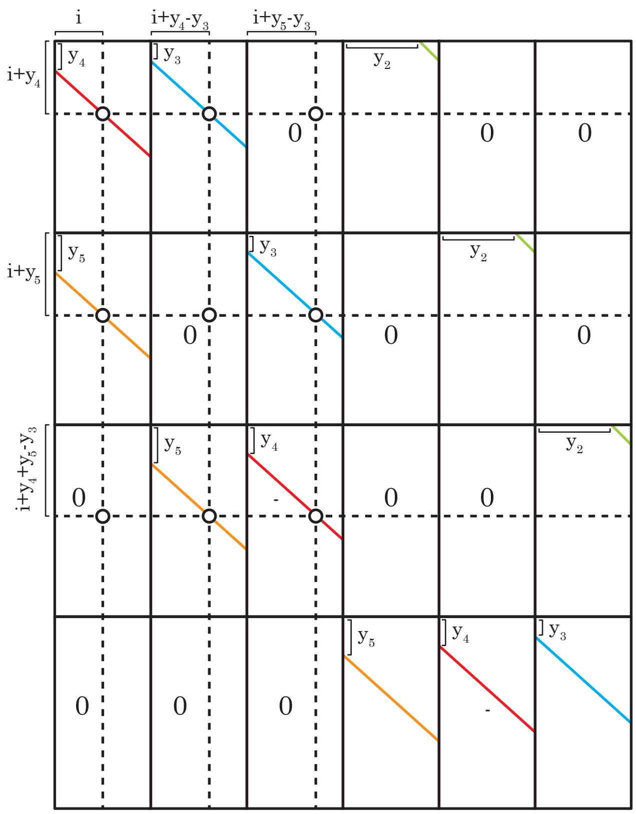

Figure 2: The size $4n \times 6y_1$ submatrix of the $4n \times 6n$ matrix.

5 Proof of Theorem 1.2

We need to find explicit p_{ij} for the sequence (1.1) and to lower bound the border ranks of $T_{k,q} = T_k|_{A^*_{2k+1-q} \otimes B^* \otimes C^*}$ as in Proposition 2.3 for all sufficiently large k and some useful $q = q(k)$. We first determine q. (Imagine our choice of q being based on the strategy of our opponent.)

Let I_q denote the index set kept to obtain $T_{k,q}$, so $|I_q| = 2k + 1 - q$. Write $I_q = I_- \sqcup I_0 \sqcup I_+$ for the negative, zero, and positive indices appearing in I_q. Note that the sets I_-, I_0, I_+ depend on q and if we want to stress this dependence we write $I_-(q)$. Fix a large constant K to be determined later. In what follows $0 < \epsilon < \frac{1}{42}$.

Setup: Let $T_{k,q}$ be a sequence of restrictions as above. Let q be the largest index such that in each of $I_-(q), I_+(q)$ there exist either

1. Three indices with absolute value at least $\frac{k}{2}$, or
2. Five indices with absolute value smaller than $\frac{k}{2}$ such that the absolute value of the difference of the smallest and largest one is at most K.

Without loss of generality we may assume that both conditions fail for $I_-(q + 1)$.

Lemma 5.1. *Let $T_{k,q}$ and q be as in the setup. Then $|I_-| \leq \frac{2k}{K} + 7$ and hence $q \geq k - (\frac{2k}{K} + 7)$.*

Proof. Partition the negative indices $\{-k, \dots, -1\}$ into groups as follows:

1. $\{-k, \dots, -\lceil\frac{k}{2}\rceil\}$
2. $\lfloor(k/2)/K\rfloor$ groups

$$\{-\lceil\frac{k}{2}\rceil + 1, \dots, -\lceil\frac{k}{2}\rceil + K\},\ \{-\lceil\frac{k}{2}\rceil + K + 1, \dots, -\lceil\frac{k}{2}\rceil + 2K\}, \dots,$$
$$\{-\lceil\frac{k}{2}\rceil + (\lceil\frac{k}{2}\rceil - 1)K + 1, \dots, -\lceil\frac{k}{2}\rceil + \lfloor(k/2)/K\rfloor K\}$$

3. $\{-\lceil\frac{k}{2}\rceil + \lfloor(k/2)/K\rfloor K + 1, \dots, -1\}$

As the conditions fail for $q + 1$ we can have at most four indices in $I_-(q + 1)$ in each of the groups in (2) or in group (3) giving at most:

$$4\frac{k}{2K} + 4 = \frac{2k}{K} + 4$$

indices in groups (2) and (3) for $I_-(q + 1)$. In the group (1) we can have at most 2 indices. The cardinality of $I_-(q)$ and $I_-(q + 1)$ may differ at most by one. The bounds follow. □

We now group the possible distributions of indices in the sets I_q into three cases and show that in each case we may obtain the desired lower bound. Note that the size of q we utilize to obtain the lower bound will depend on how the indices are removed.

Lemma 5.2. *Let $T_{k,q}$ and q be as in the setup and $K \geq \frac{78}{1-42\epsilon}$. If*

$$|I_+ \cap [0, \frac{3}{7}k]| > \frac{4}{K}\frac{3}{7}k + 4 \ \text{ or }\ |I_- \cap [-\frac{3}{7}k, 0]| > \frac{4}{K}\frac{3}{7}k + 4\,, \tag{5.1}$$

then for all $k > \frac{210K+21K\epsilon-168+56K^2}{K-78-42K\epsilon}$, one has $\underline{\mathbf{R}}(T_k) \geq (2 + \epsilon)(2k + 1)$.

Proof. Recall that $m = 2k+1$. Suppose $|I_+ \cap [0, \frac{3}{7}k]| > \frac{4}{K}\frac{3}{7}k + 4$. We partition $\{0, \ldots, \lfloor \frac{3}{7}k \rfloor\}$ into at most $\lfloor \frac{3k}{7K} \rfloor + 1$ subsets of consecutive indices of cardinality at most K. By the pigeonhole principle we may find five indices $i_1 < \cdots < i_5 \le \frac{3}{7}k$ in I_+ with $i_5 - i_1 \le K$. If we choose the smallest possible such indices, then $q \ge k - (\frac{2k}{K} + 7) + i_3 - \frac{4}{K}i_3 - 4$.

Restrict T to the subspace of $B^* \otimes C^*$ where i_3 becomes the diagonal. By Lemma 4.3 and Corollary 4.4, with $n = m - i_3$ and $C = K$, we have $\underline{\mathbf{R}}(T_{k,q}) \ge \frac{5}{3}(m - i_3) - \frac{8}{3}K$. Thus $\underline{\mathbf{R}}(T_k) \ge \frac{13}{3}k - \frac{2}{3}i_3 - \frac{4}{K}i_3 - (\frac{2k}{K} + \frac{28}{3}) - \frac{8}{3}K$. As $i_3 \le \frac{3}{7}k - 2$ we need

$$\left(\frac{13}{3} - \frac{2}{3}\frac{3}{7}\right)k + (-2 - \frac{4 \cdot 3}{7})\frac{k}{K} + \frac{8}{K} - \frac{8}{3}K - 8 \ge (2+\epsilon)(2k+1),$$

i. e.,

$$k \ge \frac{210K + 21K\epsilon - 168 + 56K^2}{K - 78 - 42K\epsilon}.$$

Now instead assume $|I_+ \cap [0, \frac{3}{7}k]| \le \frac{4}{K}\frac{3}{7}k + 4$ and thus $q \ge k - (\frac{2k}{K} + 7) + \frac{3(K-4)}{7K}k - 4$. Using this, we may conclude in the case $|I_- \cap [-\frac{3}{7}k, 0]| > \frac{4}{K}\frac{3}{7}k + 4$ by the same reasoning as above with an even better bound. ◻

In light of Lemmas 5.1 and 5.2, we may henceforth assume Eq. (5.1) fails to hold so

$$q \ge k - \left(\frac{2k}{K} + 7\right) + \frac{3}{7}k - \frac{12k}{7K} - 4 = \frac{10}{7}k - \frac{26k}{7K} - 11\,. \tag{5.2}$$

Lemma 5.3. *Let $T_{k,q}$ and q be as in the setup and assume additionally that Eq. (5.2) holds. If I_- has three indices with absolute value greater than $\frac{k}{2}$, then $\underline{\mathbf{R}}(T_k) \ge (2+\epsilon)(2k+1)$ when $K \ge \frac{39}{1-21\epsilon}$ and $k \ge \frac{(13+\epsilon)21K}{2(K-39-21K\epsilon)}$.*

Proof. Let y_3, y_4, y_5 be three indices in I_- with absolute value greater than $\frac{k}{2}$. Let y_1 and y_2 be the largest and smallest index in $I_+ \setminus [0, \frac{k}{2}]$, respectively, which exist by point (1) of the setup. Thus, we may add $\frac{k}{2} - (y_1 - y_2) - 1$ to Eq. (5.2). By Lemma 4.5 with $n = 2k+1$ we obtain:

$$\begin{aligned}\underline{\mathbf{R}}(T_k) &\ge \underline{\mathbf{R}}(T_{k,q}) + q \\ &\ge 2k + 1 + y_1 - \frac{2}{3}y_2 + \frac{10}{7}k - \frac{26k}{7K} - 11 + \frac{k}{2} - (y_1 - y_2) - 1 \\ &\ge \frac{86}{21}k - \frac{26}{7}\frac{k}{K} - 11,\end{aligned}$$

where to obtain the last line we used that $y_2 \ge \frac{k}{2}$.

This bound will be greater than $(2+\epsilon)(2k+1)$ when

$$K \ge \frac{39}{1 - 21\epsilon}$$

and

$$k \ge \frac{(13+\epsilon)21K}{2(K - 39 - 21K\epsilon)}$$

concluding the proof of Lemma 5.3. ◻

Lemma 5.4. *Let $T_{k,q}$ and q be as in the setup and assume additionally that Eq. (5.2) holds. If I_- has five indices with absolute value smaller than $\frac{k}{2}$ such that the difference of the smallest and largest is at most K, then $\underline{\mathbf{R}}(T) \geq (2+\epsilon)(2k+1)$ when $K \geq \frac{78}{1-42\epsilon}$ and $k \geq \max(\frac{21(12+\epsilon)K}{K-42\epsilon K-78}, \frac{210K+21K\epsilon-168+56K^2}{K-78-42K\epsilon})$.*

Proof. Suppose these five indices have absolute value less or equal to $\frac{3}{7}k$. Then we conclude exactly as in the first part of the proof of Lemma 5.2. Thus, we assume all indices are greater then $\frac{3}{7}k - K$.

We proceed as in Lemma 5.3. First assume there exist two indices in $I_+ \setminus [0, \frac{4}{7}k + K]$. Fix y_2 and y_1 to be, respectively, the smallest and largest index in $I_+ \setminus [0, \frac{4}{7}k + K]$, so we may add $\frac{3}{7}k - K - (y_1 - y_2)$ to Eq. (5.2).

We obtain:

$$\begin{aligned}\underline{\mathbf{R}}(T_k) &\geq \underline{\mathbf{R}}(T_{k,q}) + q \\ &\geq \left(2k + 1 + y_1 - \frac{2}{3}y_2\right) + \frac{10}{7}k - \frac{26k}{7K} - 11 + \frac{3}{7}k - K - (y_1 - y_2) \\ &\geq (4 + \frac{1}{21})k - \frac{26}{7}\frac{k}{K} - 10 - \frac{2K}{3},\end{aligned}$$

where in the last line we used that $y_2 \geq \frac{4}{7}k + K$. This is at least $(2+\epsilon)m$ when

$$K \geq \frac{78}{1 - 42\epsilon}$$

and

$$k \geq \frac{21(12 + \epsilon + \frac{2K}{3})K}{K - 42\epsilon K - 78}$$

and we conclude in this case.

If there are no two indices in $I_+ \setminus [0, \frac{4}{7}k + K]$, then by point (1) in the setup we must have two indices in $I_+ \cap [\frac{1}{2}k, \frac{4}{7}k + K]$. Let y_2 and y_1 to be, respectively, the smallest and largest in this set. We may add $\frac{1}{2}k - 1 - (y_1 - y_2)$ to Eq. (5.2). Note that $y_1 - y_2 < \frac{3}{7}k - K$. Proceeding as above obtain an even stronger lower estimate on border rank. □

Proof of Theorem 1.2. One of the cases in Lemmas 5.2, 5.3, and 5.4 must hold, which proves the theorem for k large enough.

In order for the assumptions of all lemmas to hold, we need $K \geq \frac{78}{1-42\epsilon}$ and $k \geq \frac{21(12+\epsilon+\frac{2K}{3})K}{K-42\epsilon K-78}$ and $k \geq \frac{210K+21K\epsilon-168+56K^2}{K-78-42K\epsilon}$. As $K \geq 78$, the last lower bound on k is more restrictive. Taking the estimate $K = \frac{78}{1-42\epsilon} + 1$ it holds when

$$k \geq \frac{7(52274 - 151635\epsilon + 14616\epsilon^2 + 5292\epsilon^3)}{(1 - 42\epsilon)^3}.$$ □

6 Code for the proof of Theorem 1.3

```
S=QQ[w_0..w_4];
rs=res ideal basis(1,S);
kosz=rs.dd_3;
R=QQ[a_(-6)..a_6,b_(-6)..b_6,c_(-6)..c_6,w_0..w_4]
T6:=0
for i from -6 to 6 do
(
 for j from max(-6,-i-6) to min(6,-i+6) do
  (
   T6=T6+(2^i+3^j-7*i*j)*a_i*b_j*c_(-i-j)
  );
)
by=matrix{{b_(-6)..b_6}}; cz=matrix{{c_(-6)..c_6}};
L=subsets(-6..6,4); zle={};
for m in L do (
subp={};
 for mm from -6 to 6 do(
  if (not member(mm,m)) then(
   subp=subp|{a_mm=>(sum for i from 0 to 4 list (random(0,R)*w_i))};
  );
 );
 l= rank diff(cz,diff(transpose by,diff(sub(kosz,R),sub(T6,subp))));
if (l<127) then zle=zle|{m};
)
#zle
```

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JOSEPH (JM) LANDSBERG works on geometric problems originating in complexity theory. He obtained his Ph. D. in 1990 from Duke University under the direction of Robert Bryant on minimal submanifolds and a generalization of calibrations. His past research was at the interface of algebraic geometry, differential geometry and representation theory. He is an Owen Professor of Mathematics at Texas A&M University. Landsberg has directed fifteen Ph. D. theses, is the author of five books:

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- *Tensors: Asymptotic Geometry and Developments 2016-2018*, AMS (2019) [AMS Bookstore]
- *Geometry and Complexity Theory*, Cambridge U. Press (2017) [CUP]
- *Tensors: Geometry and Applications*, AMS (2012) [AMS Bookstore]
- *Cartan for Beginners* (co-author Thomas A. Ivey), AMS (2003), 2nd edition (2016) [AMS Bookstore]

and has published over 100 research articles. Fall 2025 he co-organized the semester "Complexity and Linear algebra" at the Simons Institute for the Theory of Computing, UC Berkeley. At the same time he also served as a UC Berkeley Chancellor's Professor. Landsberg was first exposed to the beauty of geometry in college while taking a calculus course for Physics majors being taught by Joe Harris, and first exposed to complexity by Peter Bürgisser, who offered him a bottle of champagne if he could solve a question about 2×2 matrices.

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