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The Complexity of the Fermionant and Immanants of Constant Width

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Abstract: In the context of statistical physics, Chandrasekharan and Wiese recently introduced the *fermionant* Ferm_k, a determinant-like function of a matrix where each permutation π is weighted by $-k$ raised to the number of cycles in π. We show that computing Ferm_k is #P-hard under polynomial-time Turing reductions for any constant $k > 2$, and is $\oplus$P-hard for $k = 2$, where both results hold even for the adjacency matrices of planar graphs. As a consequence, unless the polynomial-time hierarchy collapses, it is impossible to compute the immanant $\mathrm{Imm}_\lambda A$ as a function of the Young diagram λ in polynomial time, even if the width of λ is restricted to be at most 2. In particular, unless NP $\subseteq$ RP, Ferm_2 is not in P, and there are Young diagrams λ of width 2 such that Imm_λ is not in P.

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1 Introduction

The permanent and determinant of a matrix have deep and well-known connections with statistical physics. Physicists are generally concerned with a type of generating function called the *partition function*,

$$Z = \sum_s \mathrm{e}^{-\beta E(s)} .$$

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If the system consists of n sites each of which has a spin that can point up or down, say, this sum ranges over all 2^n states s of the system. The summand is the Boltzmann factor, where $E(s)$ is the energy and β is the inverse temperature. Virtually any physical quantity can then be written as a derivative of Z with respect to an appropriate variable, representing the physical interactions in the system.

In a number of systems of physical interest, such as the Ising model of magnetism, Z can be rewritten as a sum over all perfect matchings of a weighted graph, where the weight of each matching is the product of the (possibly complex-valued) weights of its edges. In the bipartite case, this sum is the permanent of the weighted adjacency matrix. But in the planar case, the weights can be modified in such a way that this permanent (or in the non-bipartite case, a Pfaffian) becomes a determinant. Computing the partition function is then a simple matter of finding the eigenvalues of this matrix, and indeed this is one way to derive the celebrated exact solution of the Ising model in two dimensions [20, Ch. 13].

Inspired by these connections, Chandrasekharan and Wiese [10] recently showed that the partition functions of certain models in quantum statistical physics can be written in terms of a quantity they call the *fermionant*. Let A be an $n \times n$ matrix. The *fermionant* of A, with parameter k, is defined as

$$\mathrm{Ferm}_k A = (-1)^n \sum_{\pi \in S_n} (-k)^{c(\pi)} \prod_{i=1}^{n} A_{i,\pi(i)} .$$

Here S_n denotes the symmetric group, i. e., the group of permutations of n objects, and $c(\pi)$ denotes the number of cycles in π. This raises the interesting question of whether the fermionant can be computed in polynomial time, especially in the case $k = 2$, which corresponds to fermionic systems.

Since $(-1)^{n+c(\pi)}$ is also the parity of π, the fermionant for $k = 1$ is simply the determinant, which can of course be computed in polynomial time. But this appears to be the only value of k for which this is possible. We prove the following:

Theorem 1.1. *For any constant $k > 2$, computing* Ferm_k *for the adjacency matrix of a planar graph is #P-hard under polynomial-time Turing reductions. Moreover, unless NP ⊂ RP, for $k > 2$ there can be no fully polynomial randomized approximation scheme (FPRAS) that computes* Ferm_k *within a multiplicative error $1+\varepsilon$ in time polynomial in n and $1/\varepsilon$ with probability at least $3/4$.*

Our proof consists of a reduction to the fermionant from certain values of the Tutte polynomial for planar graphs. Theorem 1.1 then follows from Vertigan's results [24] on the #P-hardness of planar Tutte polynomials, and the recent results of Goldberg and Jerrum [14] on their inapproximability.

For the most physically relevant case $k = 2$, we have a slightly weaker result. Recall that ⊕P is the class of problems of the form "is $|S|$ odd," where S is a set such that we can tell whether $x \in S$ in polynomial time.

Theorem 1.2. *Computing* Ferm_2 *for the adjacency matrix of a planar graph is ⊕P-hard.*

By Toda's theorem [21], the polynomial-time hierarchy PH reduces to ⊕P under randomized polynomial-time reductions. Therefore, unless PH collapses, and indeed unless NP ⊂ RP, Ferm_2 is not in P.

Theorems 1.1 and 1.2 also imply new hardness results for the *immanant*. Recall that a *Young diagram* λ is an nonincreasing integer partition of n, $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_t$ such that $\sum_i \lambda_i = n$. They are often drawn as diagrams with λ_1 boxes on the first row, λ_2 boxes on the second row, and so on. The *width* of a Young diagram is λ_1, and its *depth* is the largest t such that $\lambda_t > 0$. Each Young diagram is associated with an irreducible character χ_λ of S_n, and the immanant Imm_λ of a matrix A is

$$\mathrm{Imm}_\lambda A = \sum_{\pi \in S_n} \chi_\lambda(\pi) \prod_{i=1}^{n} A_{i,\pi(i)} .$$

If χ_λ is the parity $(-1)^\pi$ or the trivial character 1, the immanant is the determinant or the permanent respectively. Thus we can think of the immanant as interpolating, in some sense, from $\det A$ to $\mathrm{perm}\, A$.

Since the determinant is in P but the permanent is #P-hard [22, 20], it makes sense to ask how the complexity of the immanant varies as λ's Young diagram ranges from a single row of length n (the trivial representation) to a single column (the parity). Strengthening earlier results of Hartmann [15], Bürgisser [8] showed that the immanant is #P-hard if λ is a hook or rectangle of polynomial width,

$$\lambda_1 = w, \lambda_2 = \cdots = \lambda_{n-w+1} = 1 \quad \text{or} \quad \lambda_1 = \cdots = \lambda_{n/w} = w ,$$

where $w = \Omega(n^\delta)$ for some $\delta > 0$. Recently Brylinski and Brylinski [7] improved these results by showing that the immanant is #P-hard whenever two successive rows have a polynomial "overhang," i. e., if there is an i such that $\lambda_i - \lambda_{i+1} = \Omega(n^\delta)$ for some $\delta > 0$. The case where λ has large width but small overhang, such as a "ziggurat" where $\lambda_i = w - i + 1$ and $n = w(w+1)/2$, remains open.

At the other extreme, Barvinok [3] and Bürgisser [9] showed that the immanant is in P if λ is extremely close to the parity, in the sense that the leftmost column contains all but $O(1)$ of the n boxes. Specifically, [9] gives an algorithm that computes $\mathrm{Imm}_\lambda A$ in time $O(s_\lambda d_\lambda n^2 \log n)$, where s_λ and d_λ are the number of standard and semistandard tableaux of shape λ respectively. Recall that a standard tableau is a labeling of the boxes of a Young tableau with the numbers $1, 2, \ldots, n$ such that each row and column is strictly increasing. In a semistandard tableau, the columns are strictly increasing and the rows are nondecreasing. If the height of λ is $n - c$ where $c = O(1)$, then s_λ and d_λ are bounded above by $\binom{n}{c}\sqrt{c!} = O(n^c)$ and $\binom{n}{c}\binom{n+c-1}{c} = O(n^{2c})$ respectively, giving a polynomial-time algorithm for Imm_λ.

Any function of the cycle structure of a permutation is a *class function*, i. e., it is invariant under conjugation. Since any class function is a linear combination of irreducible characters, the fermionant is a linear combination of immanants. If k is a positive integer, it has nonzero contributions from immanants whose Young diagrams have width k or less:

$$\mathrm{Ferm}_k A = \sum_\lambda d_\lambda^{(k)} \, \mathrm{Imm}_{\lambda^T} A . \tag{1}$$

Here λ ranges over all Young diagrams with depth at most k, $d_\lambda^{(k)}$ denotes the number of semistandard tableaux of shape λ and content in $\{1, \ldots, k\}$, and λ^T denotes the transpose of a Young diagram, i. e., its reflection around the diagonal:

$$\lambda_i^T = \max\{ j : \lambda_j \geq i \} .$$

To derive (1), first note that the class function $k^{c(\pi)}$ is the trace of π's action on $(\mathbb{C}^k)^{\otimes n}$ by permuting the coordinates of the tensor product,

$$\pi(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = v_{\pi(1)} \otimes v_{\pi(2)} \otimes \cdots \otimes v_{\pi(n)} .$$

By Schur duality (see,e. g., [13]) the multiplicity of λ in $(\mathbb{C}^k)^{\otimes n}$ is $d_\lambda^{(k)}$, so

$$k^{c(\pi)} = \sum_\lambda d_\lambda^{(k)} \chi_\lambda(\pi) .$$

We then transform $k^{c(\pi)}$ to $(-1)^n(-k)^{c(\pi)}$ by tensoring each λ with the sign representation, flipping it to its transpose λ^T, which has width at most k.

If $k = O(1)$, there are $O(n^{k-1}) = \mathrm{poly}(n)$ Young diagrams of width k or less. Thus for any constant k there is a polynomial-time Turing reduction from the fermionant Ferm_k to the problem of computing the immanant Imm_λ where λ is given as part of the input, and where λ has width at most k. Therefore, Theorems 1.1 and 1.2 give us the following corollary.

Corollary 1.3. *For any constant integer k, the problem of computing the immanant* $\mathrm{Imm}_\lambda A$ *as a function of A and* λ *is #P-hard under polynomial-time Turing reductions if* $k \geq 3$, *and* $\oplus$*P-hard if* $k = 2$, *even if* λ *is restricted to Young diagrams with width k or less.*

In particular, unless the polynomial-time hierarchy collapses, there exist diagrams λ of width 2 such that Imm_λ is not in P. This is somewhat surprising, since these immanants are "close to the determinant" in some sense.

This partly answers a question of Bürgisser [8], who asked whether immanants of width $k = 2$ are VNP-complete in the arithmetic model. Indeed, we conjecture that the fermionant is #P-hard, as opposed to just $\oplus$P-hard, when $k = 2$. Moreover, we conjecture that the immanant is #P-hard for any family of Young diagrams of depth $n - \Omega(n^\delta)$, or equivalently, any family with a polynomial number of boxes to the right of the first column:

Conjecture 1.4. *Let* $\lambda(n)$ *be any family of Young diagrams of depth* $n - \Omega(n^\delta)$ *for some constant* $\delta > 0$. *Then* $\mathrm{Imm}_{\lambda(n)}$ *is #P-hard.*

Roughly speaking, this would imply that the results of [3, 9] showing that Imm_λ is in P if λ has depth $n - O(1)$ are tight.

2 #P-hardness from circuit partitions and the Tutte polynomial

Proof of Theorem 1.1. Our proof consists of reduction from the Tutte polynomial to the fermionant, through the circuit partition polynomial. Let G be a directed graph. A *circuit partition* of G is a partition of G's edges into circuits, i. e., sets of directed edges $\{(v_1, v_2), (v_2, v_3), \ldots, (v_s, v_1)\}$. Let r_t denote the number of circuit partitions containing t circuits; for instance, r_1 is the number of Eulerian circuits. The *circuit partition polynomial* $j(G;z)$ is the generating function

$$j(G;z) = \sum_{t=1}^{\infty} r_t z^t . \tag{2}$$

This polynomial was first studied by Martin [19], with a slightly different parametrization; see also [1, 4, 5, 11, 16, 17, 18].

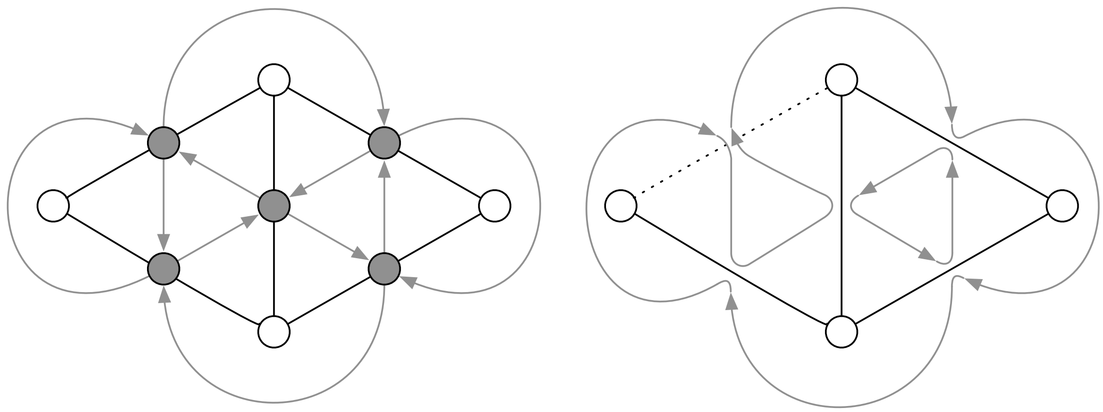

Figure 1: Left, a planar graph G (white vertices and black edges) and its medial graph G_m (gray vertices and directed edges). Right, a subset of the edges of G, and the corresponding circuit partition of G_m.

For planar graphs, $j(G;z)$ has a known relationship with the Tutte polynomial [12, 19] which we review here. Recall that the Tutte polynomial of an undirected graph $G=(V,E)$ can be written as a sum over all spanning subgraphs of G, i. e., all subsets S of E,

$$T(G;x,y)=\sum_{S\subseteq E}(x-1)^{c(S)-c(G)}\,(y-1)^{c(S)+|S|-|V|}\,. \tag{3}$$

Here $c(G)$ denotes the number of connected components in G. Similarly, $c(S)$ denotes the number of connected components in the spanning subgraph (V,S), including isolated vertices. When $x=y$,

$$T(G;x,x)=\sum_{S\subseteq E}(x-1)^{c(S)+\ell(S)-c(G)}\,, \tag{4}$$

where $\ell(S)=c(S)+|S|-|V|$ is the total excess of the components of S, i. e., the number of edges that would have to be removed to make S a forest.

If G is planar, then we can define its directed medial graph G_m. Each vertex of G_m corresponds to an edge of G, edges of G_m correspond to shared vertices in G, and we orient the edges of G_m so that they go counterclockwise around the interior faces of G, and clockwise around the outside of G (alternatively, we think of G and G_m as drawn on a sphere, in which case these edges go counterclockwise around the face consisting of the rest of the plane). Each vertex of G_m has in-degree and out-degree 2, so G_m is Eulerian. The following identity is due to Martin [19]; see also [17], or [2] for a review.

$$j(G_m;z)=z^{c(G)}\,T(G;z+1,z+1)\,. \tag{5}$$

We illustrate this in Figure 1. There is a one-to-one correspondence between subsets $S\subseteq E$ and circuit partitions of G_m. Let v be a vertex of G_m corresponding to an edge e of G. If $e\in S$, the circuit partition "bounces off e," connecting each of v's incoming edges to the outgoing edge on the same side of e. If $e \notin S$, the partition crosses over e in each direction. The number of circuits is then $c(S)+\ell(S)$, in which case (4) yields (5).

To reduce the circuit partition polynomial to the fermionant, we simply have to turn Eulerian-style circuit partitions, which cover each edge once, into Hamiltonian-style ones, which cover each vertex once. Given a directed graph G, let G_e be its line graph. Each vertex of G_e corresponds to an edge (u,v) of G, and there is an edge between (u,v) and (u',v') if $v = u'$ so that (u,v) and (u',v') could form a directed path of length two. Then each circuit partition of G corresponds to a permutation π of the vertices of G_e, and

$$\mathrm{Ferm}_k A_e = j(G;-k)\,,$$

where A_e is the adjacency matrix of G_e. In particular, if G is planar, G_m is its medial graph, $G_{m,e}$ is the line graph of G_m (which is also planar) and $A_{m,e}$ is its adjacency matrix, then

$$\mathrm{Ferm}_k A_{m,e} = (-k)^{c(G)}\,T(G;1-k,1-k)\,. \tag{6}$$

Having derived a reduction from the Tutte polynomial to the fermionant, we complete the proof by referring to known hardness results on the Tutte polynomial. Vertigan [24] proved that computing $T(G;x,y)$ for planar graphs is #P-hard under polynomial-time Turing reductions, except on the set

$$\{x,y : (x-1)(y-1) \in \{1,2\}\} \cup \{(1,1),(-1,-1),(\omega,\omega^*),(\omega^*,\omega)\}\,,$$

where $\omega = \mathrm{e}^{2\pi i/3}$ is the cube root of unity. Moreover, Goldberg and Jerrum [14] showed that $T(G;x,y)$ is inapproximable for planar graphs at many values of x and y, and in particular in the region $x,y < -1$. Then (6) implies that Ferm_k is #P-hard, and inapproximable, for any $k > 2$. □

3 ⊕P-hardness from Hamiltonian circuits

Proof of Theorem 1.2. We will prove Theorem 1.2 by showing that the fermionant Ferm_2 can be used to compute the parity of the number of Hamiltonian circuits in an undirected graph of size $n > 4$.

Let A be the adjacency matrix of an undirected graph G with n vertices and no self-loops or multiple edges. Each permutation $\pi \in S_n$ that gives a non-zero contribution to $\mathrm{Ferm}_2(A)$ corresponds to a Hamiltonian-style circuit partition of G; that is, a vertex-disjoint union of undirected cycles that includes every vertex. A 2-cycle in π corresponds to a circuit that travels back and forth along a single edge. Any circuit of length greater than 2 can be oriented in two ways, so a circuit partition with c_2 2-cycles and c' longer cycles corresponds to $2^{c'}$ permutations.

Taken together, these permutations contribute $(-1)^n\,(-2)^{c_2}(-4)^{c'}$ to $\mathrm{Ferm}_2 A$. This is a multiple of 8 unless $c' = 0$ and $c_2 \le 2$, which implies $n \le 4$, or if $c_2 = 0$ and $c' = 1$, which corresponds to a Hamiltonian cycle. In the latter case $\mathrm{Ferm}_2 A$ is a multiple of 4 but not of 8. Thus if $n > 4$ we have

$$\frac{1}{4}\,\mathrm{Ferm}_2 A \equiv_2 \#H\,, \tag{7}$$

where $\#H$ denotes the number of Hamiltonian cycles of G. Valiant [23] showed, using a parsimonious reduction from 3-SAT to HAMILTONIAN CYCLE, that the problem ⊕HAMILTONIAN CIRCUITS of computing #H mod 2 is $\oplus$P-complete. Indeed, he showed this even in the case where G is planar and every vertex has degree 2 or 3. Since (7) gives a reduction from $\oplus$HAMILTONIAN CIRCUITS to Ferm_2, this shows that Ferm_2 is $\oplus$P-hard as well. □

More generally, Ferm_k is at least as hard as computing the number of Hamiltonian circuits mod k. But for $k \geq 3$, our #P-hardness result from Theorem 1.1 is stronger.

4 Conclusion

Is the fermionant #P-hard for $k = 2$? And are immanants #P-hard even for some Young diagrams of width 2, as in Conjecture 1.4? For instance, is Imm_λ hard for the Young diagram λ of width 2 and height $n/2$?

The reduction of Theorem 1.1 fails in this case $k = 2$, since for $x = y = -1$ the Tutte polynomial is easy to compute [24]. In particular, if $G = (V,E)$ then

$$\mathrm{Ferm}_2 A_{m,e} = (-2)^{c(G)}\, T(G;-1,-1) = (-2)^{c(G)}\, (-1)^{|E|}\, (-2)^{\dim B}\,.$$

Here $\dim B$ is the dimension of G's *bicycle space*—the set of functions from E to $\mathbb{Z}_2$ that can be written both as linear combinations of cycles and as linear combinations of stars—which we can determine in polynomial time using linear algebra.

However, there is no reason to think that we might not be able to prove #P-hardness for Ferm_2 using another reduction. After all, the number of Eulerian circuits of the medial graph G_m is the number of spanning trees of G, which is also in P—but Brightwell and Winkler showed that counting Eulerian circuits is #P-hard in general [6].

Finally, it is interesting that our proof of #P-hardness is an indirect reduction from, say, the chromatic polynomial, passing through the Tutte polynomial and its ability to perform polynomial interpolation (under polynomial-time Turing reductions) by decorating the graph. Thus another open question is whether there is a more direct reduction to the fermionant from the permanent or, say, #HAMILTONIAN CIRCUITS.

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