A Comment on the LP Relaxation for the Asymmetric Traveling Salesman Path Problem

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Abstract

We observe that the LP relaxation for the Asymmetric Traveling Salesman Path Problem suggested in Section 5 of the paper "*An $O(\log n)$ Approximation Ratio for the Asymmetric Traveling Salesman Path Problem*" by Chekuri and Pál [1] is not accurate, and state a corrected linear relaxation for the problem. The inaccuracy occurs in the statement of an open problem and does not affect the validity of any of the results in [1].

An asymmetric metric (V, l) on vertex-set V is a function $l : V \times V \to \mathbb{R}^+$ that satisfies the triangle inequality: $l(u, w) \leq l(u, v) + l(v, w)$ for all $u, v, w \in V$. The Asymmetric Traveling Salesman Path Problem (ATSPP) is defined as follows: given an n-vertex asymmetric metric (V, l) and a pair of vertices $s, t \in V$, find an $s - t$ path of minimum length that visits all vertices in V. The following linear programming relaxation for ATSPP was suggested in [1], and the authors asked whether its integrality gap is bounded by $O(\log n)$. In the following, A denotes the set of all arcs in the complete digraph on vertex-set V, and for any set $S \subseteq V$, $\delta^-(S) = \{(u, v) \in A \mid u \notin S, v \in S\}$ and $\delta^+(S) = \{(u, v) \in A \mid u \in S, v \notin S\}$.

$$
\text{(ATSPP-LP)} \qquad
\begin{aligned}
&\min \sum_{a \in A} l(a)x(a) && \\
&\sum_{a \in \delta^-(v)} x(a) = 1 && \forall v \in V \setminus \{s\} \\
&\sum_{a \in \delta^+(v)} x(a) = 1 && \forall v \in V \setminus \{t\} \\
&\sum_{a \in \delta^-(S)} x(a) \geq 1 && \forall S \subseteq V \setminus \{s\} \\
&\sum_{a \in \delta^+(S)} x(a) \geq 1 && \forall S \subseteq V \setminus \{t\} \\
&x(a) \geq 0 && \forall a \in A
\end{aligned}
$$

This is clearly a relaxation of ATSPP. However, even the integer program corresponding to ATSPP-LP, where the arc variables $x(a)$ are constrained to be in $\{0, 1\}$, can have an optimal value that is smaller than the optimal solution to ATSPP by a factor of $\Omega(n)$. This can be seen from the

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following example. The asymmetric metric (V, l) in the example is the shortest path metric induced by the arc-weighted digraph G in Figure 1. Graph G is defined on vertices $V = \{s, t, v_1, \cdots, v_{n-2}\}$ and arcs $E = \{(v_i, s) \mid 1 \le i \le n-2\} \cup \{(t, v_i) \mid 1 \le i \le n-2\} \cup \{(s,t)\}$; The length of all arcs in $E \setminus \{(s,t)\}$ is zero and arc (s,t) has length 1.

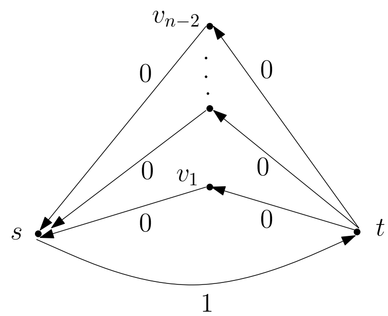

Figure 1: The directed graph G in the example, with arc lengths.

It is clear that the minimum length $s-t$ path in metric (V, l) that visits all vertices has length $n-1$; so the optimal value of the ATSPP instance is $n-1$. On the other hand, setting $x(a) = 1$ for all $a \in E$ and $x(a) = 0$ otherwise, we obtain a feasible solution to ATSPP-LP; so the optimal value of the linear program ATSPP-LP is 1. In fact, this shows that even the integer program corresponding to ATSPP-LP has optimal value 1. In this example, the ratio of the optimal value of ATSPP to that of ATSPP-LP is $n-1$. So the integrality gap of ATSPP-LP is $\Omega(n)$.

The trouble with the linear program ATSPP-LP is that the integer program corresponding to it is not an exact formulation of ATSPP. This can be corrected by the addition of the following two constraints to ATSPP-LP: $\sum_{a\in\delta^-(s)} x(a) = 0$ and $\sum_{a\in\delta^+(t)} x(a) = 0$. It is easy to see that with this modification, the corresponding integer program is an exact formulation of ATSPP. The corrected LP relaxation is as follows.

$$
\begin{aligned}
\min \sum_{a\in A} l(a)x(a) & & \\
\sum_{a\in\delta^-(v)} x(a) = 1 & & \forall v \in V \setminus \{s\} \\
\sum_{a\in\delta^+(v)} x(a) = 1 & & \forall v \in V \setminus \{t\} \\
\sum_{a\in\delta^-(S)} x(a) \ge 1 & & \forall S \subseteq V \setminus \{s\} \\
\sum_{a\in\delta^+(S)} x(a) \ge 1 & & \forall S \subseteq V \setminus \{t\} \\
\sum_{a\in\delta^-(s)} x(a) = \sum_{a\in\delta^+(t)} x(a) = 0 & & \\
x(a) \ge 0 & & \forall a \in A
\end{aligned}
$$

As mentioned in Chekuri and Pál [1], it is not clear whether an augmentation lemma (similar to Lemma 3.1 in [1]) can be proved relative to the optimal solution to such a linear program. Determining if the integrality gap of this LP relaxation is $O(\log n)$ is an interesting open question.

References

[1] Chandra Chekuri and Martin Pál: An $O(\log n)$ Approximation Ratio for the Asymmetric Traveling Salesman Path Problem. *Theory of Computing* **3** (2007), 197–209.